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Dynamics of compact objects clusters: A post-Newtonian study

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ABSTRACT
Compact object clusters are likely to exist in the centre of some galaxies because of mass segregation. The high densities and velocities reached in them deserves a better understanding. The formation of binaries and their subsequent merging by gravitational radiation emission is important to the evolution of such clusters. We address the evolution of such a system in a relativistic regime. The recurrent mergers at high velocities create an object with a mass much larger than the average. For this aim we modified the direct NBODY6++ code to include post-Newtonian effects to the force during two-body encounters. We adjusted the equations of motion to include for the first time the effects of both periastron shift and energy loss by emission of gravitational waves and so to study the eventual decay and merger of radiating binaries. The method employed allows us to give here an accurate post-Newtonian description of the formation of a run-away compact object by successive mergers with surrounding particles, as well as the distribution of characteristic eccentricities in the events. This study should be envisaged as a first step towards a detailed, accurate study of possible gravitational waves sources thanks to the combination of the direct NBODY numerical tool with the implementation of post-Newtonian terms on it.

Key words: black holes, N-body simulation, star clusters

1 INTRODUCTION

It is nowadays well established that most, if not all, galaxies harbour a supermassive black hole (SMBH) in their centre with a mass of some $10^{6-9} M_{\odot}$ (see e.g. the recent review by Ferrarese & Ford 2005; Ferrarese et al. 2001; Kormendy & Gebhardt 2001). There are also signs for masses of $10^{6} M_{\odot}$ (Greene & Ho 2005). In the case of our Galaxy this is even imperative; an SMBH with a mass of about $\sim 3-4\times 10^6 M_{\odot}$ (Eckart et al. 2002; Ghez et al. 2000, 2003; Schödel et al. 2002) must be ensconced in its centre. If one extends the correlation between the SMBH mass and the stellar velocity dispersion of the bulge of the host galaxy (the $\mathcal{M}_{\bullet}-\sigma$ correlation) observed for galactic nuclei (Gebhardt et al. 2000; Ferrarese & Merritt 2000) to smaller systems, like globular clusters, one should expect intermediate mass black holes (IMBH) with masses of between $10^3-10^4 M_{\odot}$ to be lurking in the centres of such stellar clusters. There are observations of M15 in the Milky Way or G1 in M31 (Gerssen et al. 2002; Gebhardt et al. 2002; van der Marel et al. 2003) which are compatible with this possibility, but $N-$body models of these clusters have been made which do not require the presence of an IMBH (Baumgardt et al. 2003b).

The densities observed in the central region of galaxies, where these very massive objects are located, are very high and may even exceed the core density of globular clusters by a factor hundred (about $10^7-10^8 M_{\odot}\ \mathrm{pc}^{-3}$ for the Galactic Centre, for instance) and thus make them very special laboratories for stellar dynamics.

On the other hand, it is not strictly excluded that the central mass concentrations are not massive black holes (MBHs). Mass segregation creates a flow of compact objects like neutron stars or stellar black holes to the central parts of the cluster (Lee 1987; Miralda-Escudé & Gould 2000) and may constitute there a cluster. This could mimic the effect of the MBH, and thus give an alternative explanation of the properties of clusters that have gone core-collapse, like M15 and G1 (Gebhardt et al. 2002; Baumgardt et al. 2003a,b; van der Marel et al. 2003). On the other hand, MBHs are favoured in the case of galaxies, in particular the Milky Way (Maoz 1998; Miller 2005).

For the case of a globular cluster it has been studied that stellar black holes are probably ejected from the system. Stellar black holes should form three body binaries and kick eachother out of the cluster (Phinney & Sigurdsson 1991; Kulkarni et al. 1993; Sigurdsson & Hernquist 1993; Portegies Zwart & McMillan 2000). Nonetheless, if the velocity dispersion is high enough, then binaries will not be created due to three body encounters, as in the classical case considered before, but to gravitational waves emission during two-body encounters. A simple way to understand this is that the components of a binary merge before a third particle or a second binary comes in sufficiently close to interact with them so as to eject the binary or one of its. Thus, ejections cannot happen in such a scenario. As a matter of fact, for velocity dispersions of

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$\gtrsim$ 300 km/s the merging time in clusters with two mass components is already shorter than the required time between interactions before a third particle or a second binary can bring about an ejection (Lee 1995).

Relativistic stellar dynamics is of paramount importance for the study of a number of subjects. For instance if we want to have a better understanding of what the constraints on alternatives to supermassive black holes are; in order to canvass the possibility of ruling out stellar clusters, one must do detailed analysis of the dynamics of relativistic clusters and determine in particular the core collapse time (Miller 2005). Also, when we want to more competently dive into the formation of MBHs, learn how the dynamics around them is, for instance to estimate captures of compact objects on a central SMBH via extreme mass ratio inspiraling, or peruse a system of many supermassive black holes etc the inclusion of relativistic effects is constitutive.

Our current work includes the study of stars on relativistic orbits around a SMBH, so as to be able to estimate captures of compact objects on a central SMBH via extreme mass ratio inspiraling and binary evolution of two SMBHs.

Efforts to understand the dynamical evolution of a stellar cluster in which relativistic effects may be important have been already done by Lee (1987), Quinlan & Shapiro (1989), Quinlan & Shapiro (1990) and Lee (1993). In his work, Lee (1993) (MHL93 from now onwards) addressed the problem of the dynamical evolution of a cluster composed of compact objects by, with some approximations, adding an estimate of the gravitational wave emission term correction to NBODY5 (see section 3). Nevertheless, he neglected the $1\mathcal{PN}$ and $2\mathcal{PN}$ terms and made use of the formalism introduced by Peters (1964), possibly because of computational bourns. The computation of the $\mathcal{PN}$ corrections is CPU-consuming, for we have to compute both, the accelerations and their time-derivatives (see next section). Also, NBODY5 is not suitable for supercomputers or special purpose GRAPE hardware; here either NBODY6++ or NBODY4 have to be used (Spurzem 1999; S. J. Aarseth 1999).

In this work we describe a new tool that allows us to address this problem in a much more rigurous way than done in the existing literature, including deviations from the Newtonian formalism of the standard direct NBODY6++ code (Spurzem 1999), based on Aarseth's direct NBODY codes (S. J. Aarseth 1999). We modified it in order to allow for post-Newtonian ($\mathcal{PN}$) effects, implementing in it the $1\mathcal{PN}$, $2\mathcal{PN}$ and $2.5\mathcal{PN}$ corrections without any further approximation than those indwelling to the calculation of the $\mathcal{PN}$ terms themselves (Soffel 1989).

In Section 2 we give a brief description of the method and of the implementation of the $\mathcal{PN}$ terms into a standard NBODY code. An analysis of the formation and evolution of a particle that gains more and more mass from successive mergers in the system (the "runaway particle") is made in Section 3 and, to conclude, in Section 4 we make a summary and discussion of the main results obtained.

2 METHOD: DIRECT SUMMATION NBODY WITH POST-NEWTONIAN CORRECTIONS

The version of direct summation NBODY method we employed for the calculations, NBODY6++, includes the *KS regularisation*. This means that when two particles are tightly bound to each other or the separation among them becomes smaller during a hyperbolic encounter, the couple becomes a candidate for a in order to avoid problematical small individual time steps (Kustaanheimo and Stiefel 1965). We modified this scheme to allow for relativistic corrections to the Newtonian forces by expanding the acceleration in a series of powers of $1/c$ in the following way (Damour & Dereulle 1981; Soffel 1989):

$$\underline{a} = \underbrace{\underline{a}_0}_{\text{Newt.}} + \underbrace{\underbrace{c^{-2}\underline{a}_2}_{1\mathcal{PN}} + \underbrace{c^{-4}\underline{a}_4}_{2\mathcal{PN}}}_{\text{periastron shift}} + \underbrace{\underbrace{c^{-5}\underline{a}_5}_{2.5\mathcal{PN}}}_{\text{grav. rad.}} + \mathcal{O}(c^{-6}), \tag{1}$$

where $\underline{a}$ is the acceleration of particle 1, $\underline{a}_0 = -Gm_2\underline{n}/r^2$ is the Newtonian acceleration, G is the gravitation constant, m_1 and m_2 are the masses of the two particles, r is the distance of the particles, $\underline{n}$ is the unit vector pointing from particle 2 to particle 1, and the $1\mathcal{PN}$, $2\mathcal{PN}$ and $2.5\mathcal{PN}$ are post-Newtonian corrections to the Newtonian acceleration, responsible for the pericenter shift ($1\mathcal{PN}$, $2\mathcal{PN}$) and the quadrupole gravitational radiation ($2.5\mathcal{PN}$), correspondingly, as shown in Eq. (1). The expressions for the accelerations are:

$$\underline{a}_2 = \frac{Gm_2}{r^2}\Big\{\underline{n}\Big[-v_1^2 - 2v_2^2 + 4v_1v_2 + \frac{3}{2}(nv_2)^2 + 5\big(\frac{Gm_1}{r}\big) + 4\big(\frac{Gm_2}{r}\big)\Big] + (\underline{v}_1 - \underline{v}_2)\big[4nv_1 - 3nv_2\big]\Big\} \tag{2}$$

$$\begin{aligned}\underline{a}_4 = \frac{Gm_2}{r^2}\Big\{&\underline{n}\Big[-2v_2^4 + 4v_2^2(v_1v_2) - 2(v_1v_2)^2 \\ &+ \frac{3}{2}v_1^2(nv_2)^2 + \frac{9}{2}v_2^2(nv_2)^2 - 6(v_1v_2)(nv_2)^2 \\ &- \frac{15}{8}(nv_2)^4 + \big(\frac{Gm_1}{r}\big)\Big(-\frac{15}{4}v_1^2 + \frac{5}{4}v_2^2 - \frac{5}{2}v_1v_2 \\ &+ \frac{39}{2}(nv_1)^2 - 39(nv_1)(nv_2) + \frac{17}{2}(nv_2)^2\Big) \\ &+ \big(\frac{Gm_2}{r}\big)(4v_2^2 - 8v_1v_2 + 2(nv_1)^2 \\ &- 4(nv_1)(nv_2) - 6(nv_2)^2)\Big] \\ &+ (\underline{v}_1 - \underline{v}_2)\Big[v_1^2(nv_2) + 4v_2^2(nv_1) - 5v_2^2(nv_2) \\ &- 4(v_1v_2)(nv_1) + 4(v_1v_2)(nv_2) - 6(nv_1)(nv_2)^2 \\ &+ \frac{9}{2}(nv_2)^3 + \big(\frac{Gm_1}{r}\big)\big(-\frac{63}{4}nv_1 + \frac{55}{4}nv_2\big) \\ &+ \big(\frac{Gm_2}{r}\big)\big(-2nv_1 - 2nv_2\big)\Big]\Big\} \\ &+ \frac{G^3m_2}{r^4}\underline{n}\big[-\frac{57}{4}m_1^2 - 9m_2^2 - \frac{69}{2}m_1m_2\big],\end{aligned} \tag{3}$$

$$\begin{aligned}\underline{a}_5 = \frac{4}{5}\frac{G^2m_1m_2}{r^3}\Big\{&(\underline{v}_1 - \underline{v}_2)\Big[-(\underline{v}_1 - \underline{v}_2)^2 + 2\big(\frac{Gm_1}{r}\big) - 8\big(\frac{Gm_2}{r}\big)\Big] \\ &+ \underline{n}(nv_1 - nv_2)\big[3(\underline{v}_1 - \underline{v}_2)^2 - 6\big(\frac{Gm_1}{r}\big) + \frac{52}{3}\big(\frac{Gm_2}{r}\big)\big]\Big\}.\end{aligned} \tag{4}$$

In the last expressions $\underline{v}_1$ and $\underline{v}_2$ are the velocities of the particles. For simplification, we have denoted the vector product of two vectors, $\underline{x}_1$ and $\underline{x}_2$, like x_1x_2. The basis of direct NBODY4 and NBODY6++ codes relies on an improved Hermit integrator scheme (Makino & Aarseth 1992; S. J. Aarseth 1999) for which we need not only the accelerations but also their time derivative. These

derivatives are not included in these pages for succinctness. We integrated our correcting terms into the *KS regularisation* scheme as perturbations, similarly to what is done to account for passing stars influencing a KS pair. Note that formally the perturbation force in the KS formalism does not need to be small compared to the two-body force (Mikkola 1997). If the internal KS time step is properly adjusted, the method will work even for relativistic terms becoming comparable to the Newtonian force component.

3 DYNAMICAL EVOLUTION OF A CLUSTER OF COMPACT OBJECTS

3.1 The initial system and units

The units used in our models correspond to the so-called N-body unit system, in which $G = 1$, the total initial mass of the stellar cluster is 1 and its initial total energy is $-1/2$ (Hénon 1971; Heggie & Mathieu 1986). The system was chosen to be initially to be indentical to that employed by Lee (1993); i.e. a spherical cluster with a number of compact stars $\mathcal{N}_\star = 10^3$ of identical mass m. These were distributed in an isotropic Plummer sphere, which means that the phase-space distribution function is proportional to $|E|^{7/2}$, where E is the energy per unit mass of one star. The density profile is thus $\rho(r) = \rho_0\big(1+(r/R_{\rm Pl})^2\big)^{-5/2}$, where $R_{\rm Pl}$ is the Plummer scaling length. For such a model the N-body length unit is $\mathcal{U}_{\rm l} = 16/(3\pi)\,R_{\rm Pl}$.

In the situations considered here, the evolution of the cluster is driven by 2-body relaxation. A natural time scale is the (initial) *half-mass relaxation time*. We use the definition of Spitzer (1987),

$$T_{\rm rh}(0) = \frac{0.138N}{\ln\Lambda}\left(\frac{R_{1/2}^3}{G\mathcal{M}_{\rm cl}}\right)^{1/2}. \qquad (5)$$

For instance, for a Plummer model, the half-mass radius is $R_{1/2} = 0.769\,\mathcal{U}_{\rm l} = 1.305\,R_{\rm Pl}$. $\mathcal{M}_{\rm cl}$ is the total stellar mass and $\ln\Lambda = \ln(\gamma N)$ is the Coulomb logarithm.

For the situation considered in this work, the square ratio of the central velocity dispersion $\sigma_{\rm cen}$ to the speed of light c,

$$\left(\frac{\sigma_{\rm cen}}{c}\right)^2 \approx \frac{G\mathcal{M}_{\rm cl}}{\mathcal{R}_{\rm cl}c^2} \approx \frac{\mathcal{R}_{\rm Schw}^{\rm cl}}{\mathcal{R}_{\rm cl}} \qquad (6)$$

is big enough, so that we can expect that relativistic effects play a noticeable role in the evolution of the system. For this aim, we chose $\sigma_{\rm cen}$ to be $\sim$ 4300 km/s. G is the gravitational constant, $\mathcal{R}_{\rm cl}$ is the radius of the cluster and $\mathcal{R}_{\rm Schw}^{\rm cl} = 2G\mathcal{M}^{\rm cl}/c^2$ is the Schwarzschild radius of the cluster.

In our calculations the $\mathcal{PN}$ terms are acting all the time during the calculations but obviously become important only when velocities are high. Our criterion for particles to merge is that they reach their common Schwarzschild radii $\mathcal{R}_{\rm Schw}$; i.e. the sum of their Schwarzschild radii. This is of course approximative because the $\mathcal{PN}$ treatment breaks down when particles are that close (and $v \sim c$), but this should not matter, for the merging phase is *much* faster than any stellar dynamical time. The gravitational recoil, the expected lose linear momentum in asymmetric systems in which the merger remnant receives a kick from the gravitational waves emission obviously does not show up in our models, because we truncate the series at $\mathcal{O}(c^{-5})$ and it is only to be treated as an effect of higher-order terms.

3.2 Formation of a run-away body

Even though we started with a single-mass stellar system, the masses of some objects in the cluster increased by relativistic mergers. In Fig. (1) we survey the time evolution of the mass increase. We find a number of mergers that lead to the variation of the initial single mass situation. The particle masses increase after the relativistic merging events, since we are assuming that the particles merge perfectly when they reach the distance of their $\mathcal{R}_{\rm Schw}$ (see above). We find the formation of a runaway particle that reaches almost six percent of the initial total mass by the end of the simulation, see Fig. (1). We denoted the mass of runaway body by red crosses and the mass of other mergers by blue crosses.

One can observe that the runaway body dominates the system after its fast growing phase around 300 time units, which is approximately the moment at which the core collapse of the system happens, as we can see in Fig. (2). Only some merger events which are independent from the runaway body can occur after this phase. This fast growing phase occurs at the core collapse of the system (Meylan & Heggie 1997). In Fig. (2) we follow the evolution of the so-called Lagrangian radii of the system, spheres containing 1%, 5%, 10%, 20%, 30%, 50%, 70%, 90% and 100% of the total mass of the cluster; the centre of the cluster is defined to be the centre of the mass density. Since the runaway particle is included, and in the end its mass reaches 5% of the total initial mass of the cluster, the curves corresponding to 1% and 5% roughly correspond to its evolution. We observe that the runaway stops the core collapse and allows for a expansion.

The process of mergers translates directely into a production of energy in the central regions of the cluster. The centre adapts to supply the cluster with the same amount of energy that it can obtain via relaxation, and this amount is determined by the large-scale structure.

According to, for instance, the table given in Freitag & Benz (2001), the standard value for the core collapse time is of roughly $\sim 15-20$ times the half-mass relaxation time $T_{\rm rh}$. We find nevertheless that the core collapse time is $t_{\rm cc} \sim 11T_{\rm rh}$, with a value of $\gamma = 0.11$ in the Coulomb logarithm (Giersz & Heggie 1994), which clearly suggests that the $\mathcal{PN}$ terms accelerate the collapse. This can be seen more clearly in Fig. (2), which corresponds to the same simulation but without making use of relativistic corrections. There we can see that $t_{\rm cc} \sim 380 \sim 14T_{\rm rh}$.

In Fig. (3) we show the evolution of the the runway particle mass normalised to the mass contained in the core of the cluster, defined as in Casertano, S. & Hut, P. (1985). The mass of the runway particle can grow only up to the core mass. The core mass continuously decreases as the core collapse proceeds. We see this in the figure, where the runaway particle grows and saturates to the core mass after ~ 1200 time units.

The evolution and formation of the runaway particle mass is not as fast as it was in MHL93, as we can see in his Figure 5. For our simulation the sudden jump in the growth of the mass comes in slightly later and is smoother, reaching final values for the runaway particle mass of about three times smaller than in MHL93. The reasons for the differences are to be attributed to the following: MHL93 calculated the influence of the $2.5\mathcal{PN}$ term on the orbits in an unperturbed pair and made them merge after a *decay timescale*, following the Peters (1964) formalism. This requires the assumption that particles move along their orbits on an ellipsis, only valid when they are very far from the relativistic regime. On the other hand, we implemented the $2.5\mathcal{PN}$ term in the code itself, so that the relativistic corrections are a natural feature whose influence on

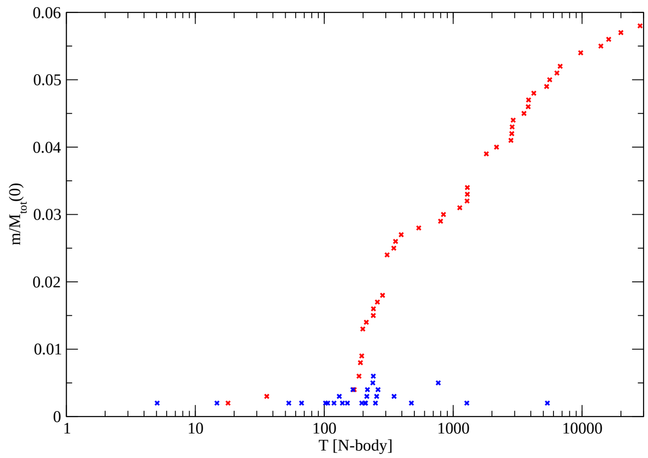

Figure 1. Time evolution of merging masses. The formation of the runaway particle is about the time of the cluster core collapse. Fore more details see text

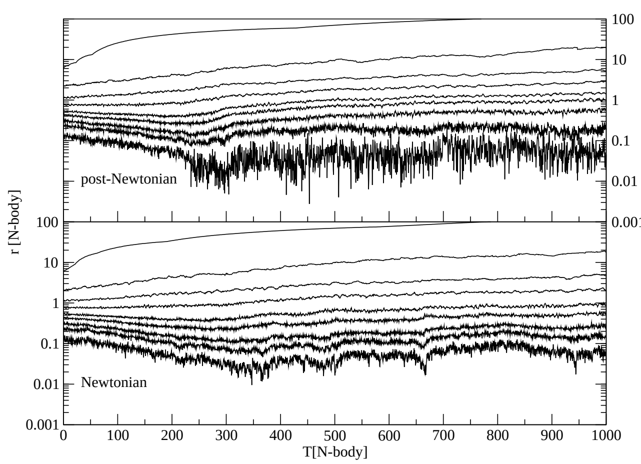

Figure 2. Evolution of the Lagrangian radii corresponding from the bottom to the top to 1%, 5%, 10%, 20%, 30%, 50%, 70%, 90% and 100% of the total mass

the evolution of the system comes in when the velocities of the stars become high enough. The influence of the $1\mathcal{PN}$ and $2\mathcal{PN}$ terms, corresponds to the conservative phase evolution of the orbit and cannot be relevant because they do not change its energy and angular momentum.

4 CONCLUSIONS

In this work we have presented a study of the formation and evolution of a runaway particle in a dense cluster of compact objects -which initially had the same mass- as a result of relativistic mergers. We employed a modified version of the direct summation NBODY6 code in which we have implemented the $1\mathcal{PN}$, $2\mathcal{PN}$, and $2.5\,\mathcal{PN}$ terms to take into account post-Newtonian corrections to the standard NBODY Newtonian acceleration.

The runaway particle reaches in the end of our simulations $\sim 6\%$ of the initial total stellar mass of the cluster. We have also compared our work to a previous result based on a more approximative scheme, the approach described in Peters (1964) and we have found out that the net result is that the growth of the runaway particle in the study of MHL93 is ~ 3 times larger. Since the

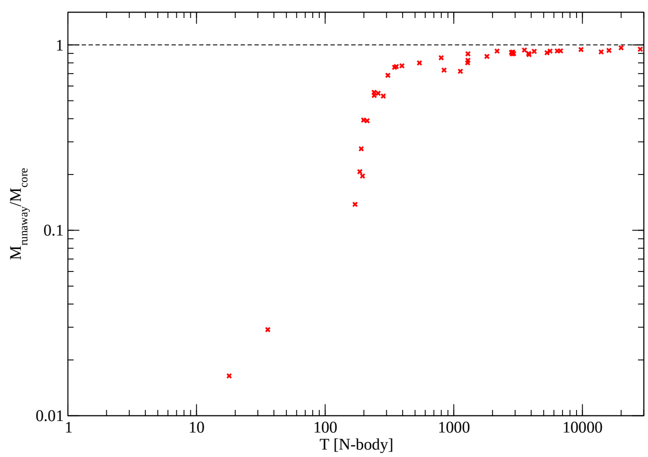

Figure 3. Evolution of the runaway particle mass in units of the core mass (at the same time)

$1\,\mathcal{PN}$, $2\,\mathcal{PN}$ terms modify the extrinsic features of the orbits (e.g. the orientation) but do not affect their intrinsic parameters (like frequency), we therefore can expect their effect to be averaged out during the evolution of the system and not influence the mergers rates. One should thus attribute the differences to the approach he made, somehow inadequate for the velocity regime considered.

This study should be envisaged as successful test test of the code, which showed to be robust. This tool can be applied to other astrophysical scenarios which require a post-Newtonian treatment. This includes on-going work, as e.g. captures of compact objects by a supermassive black hole in a galactic centre, also known as extreme mass ratio inspirals (EMRIs). One of the fundamental aims is to rigurously explore the parameter space, so that we can provide the LISA data analysis community with realistic estimates of, for instance, the eccentricity, mass ratio etc at the beginning of the final merger, when the smaller compact object enters the LISA band. An assumption for the initial parameter space is necessary in order to develop waveform "banks" for this kind of events. One must note here that the inclusion of the $1\,\mathcal{PN}$ and $2\,\mathcal{PN}$ terms is very relevant, for ressonant relaxation (or Kozai) effects, which may increase the rate of inspiral significantly, may be strongly affected by by relativistic precession and thus have an impact on the number of captures (Hopman & Alexander 2006; Kozai 1962). The inclusion of higher-order $\mathcal{PN}$ terms is also part of current study and will also shed light on other aspects of this subject (spin-spin coupling, spin-orbit interaction and radiation recoil).

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